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NARROWING OF THE REGION OF SEARCH IN PROBLEMS OF OPTIMAL SYNTHESIS OF LAYERED STRUCTURES WITH A GIVEN SET OF PROPERTIES

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The process of optimal design can be imagined as the process of diminishing the set of admissible variants. The use of necessary optimality conditions based on nonlocal variations of control parameters allows one effectively to take into account the entire set of variables that determine the structure of a layered system: the physical properties of the materials of the layers, the layer thickness, and also the total thickness of the system of layers [1–3]. To find all variants of layered structures that realize the limiting possibilities, it is necessary to distinguish the entire set of solutions that satisfy the necessary optimality conditions. The effectiveness of distinguishing these solutions depends on the extent to which the dependence of the quality functional on the set of control parameters is of a multiextremal character. The results of computational experiments show that, in problems of synthesis of layered structures under wave actions, the number of variants that satisfy the necessary optimality conditions is rather considerable. Therefore, the choice of all layered structures that lead to limiting possibilities involves significant difficulties.

In this connection, we adopt another method of studying the limiting possibilities. It is concerned with studying the existence of internal symmetry in the interrelationship of parameters that constitute an optimal structure. The existence of such symmetry in synthesis problems indicates that structures that realize limiting possibilities will group only within a certain narrow compact set Q.

The internal symmetry in the relationship of elements of the system can lead to the fact that the structures that realize limiting possibilities will satisfy additional constraints. Finding such constraints allows one to considerably decrease the dimension of the problem, because structures that realize limiting possibilities may, in addition, satisfy a certain system of m equations: $M_j(u^* = 0)$. The set of solutions of this system is the desired compact set: $Q = \{u : M_j(u) = 0,\ j = \overline{1,m}\}$.

In some cases, finding such a system of equations allows one to completely solve the synthesis problem. The main problem here is to develop an analytic procedure for describing the boundaries of the compact set being studied.

Therefore, it is of interest to study synthesis problems for layered systems in which structures that realize the limiting possibilities of controlling wave-field parameters exhibit internal symmetry. The investigation of the possibility of distinguishing a narrow compact set Q that contains the entire set of variants realizing the limiting possibilities leads to a qualitatively new method of decreasing the set of admissible structures and developing effective synthesis methods

We study three types of synthesis problems.

1. Oblique Incidence of an Electromagnetic Wave with Horizontal Polarization on a System of Nonabsorbing Magnetodielectric Layers. The propagation of an electromagnetic wave in a layered structure can be described by the following boundary-value problem:

$$\dot{f}(z) = \mu(z)g(z), \quad \dot{g}(z) = -k_0^2(\omega)u(z)f(z), \quad 0 \leqslant z \leqslant l,$$
$$g(0) = ik_{\text{low}}(\omega)\cos\theta_0(2 - f(0)), \quad g(l) = ik_{\text{upper}}(\omega)\cos\theta_{\text{upper}}f(l). \tag{1}$$

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Here $f(z)$ $(0 \leqslant z \leqslant l)$ is the complex amplitude of the electromagnetic wave, θ_0 is the angle of incidence of the electromagnetic wave, θ_{upper} is the angle at which the electromagnetic wave leaves the layered structure: $\cos\theta_{\mathrm{upper}} = (1 - (\varepsilon_{\mathrm{low}}/\varepsilon_{\mathrm{upper}})\sin^2\theta_0)^{1/2}$, $k_0 = \omega/c$ is the wavenumber in vacuum, c is the speed of light, $k_{\mathrm{low}}(\omega) = (\omega/c)\sqrt{\varepsilon_{\mathrm{low}}}$, $k_{\mathrm{upper}}(\omega) = (\omega/c)\sqrt{\varepsilon_{\mathrm{upper}}}$, $u(z) = (\varepsilon(z)\mu(z) - \varepsilon_{\mathrm{low}}\sin^2\theta_0)/\mu(z)$, $\varepsilon(z)$, $\mu(z)$ $(0 \leqslant z \leqslant l)$ is the distribution of the permittivity and permeability across the thickness of the layered structure, and $\varepsilon_{\mathrm{low}}$ and $\varepsilon_{\mathrm{upper}}$ are the permittivities of half-spaces that border the layered structure. The physical parameters of the layered structure are related by the functional dependence $\mu = \mu(\varepsilon)$, which allows one to determine uniquely the permeability of the admissible material from its known permittivity. In this case, the permittivity ε will be the only independent physical parameter. Let the admissible set consist of two materials. For each $z \in [0, l]$, the following inclusion holds:

$$\varepsilon(z) \in \Lambda. \tag{2}$$

It is required to design a layered structure with extreme spectral characteristic, whose parameters lead to a minimum value of the quality criterion

$$J = \int\limits_{\omega_{\min}}^{\omega_{\max}} \tau(\omega)\mathcal{T}(\omega)\,d\omega \tag{3}$$

under the extra condition that, at a certain frequency $\omega = \omega^*$, the functional characteristic of the design structure should be maximal:

$$\mathcal{T}(\omega^*) = \mathcal{T}^*_{\omega^*}(\omega^*). \tag{4}$$

Here $\mathcal{T}(\omega)$ is the energy transmission factor for the electromagnetic wave:

$$\mathcal{T}(\omega) = \frac{\cos\theta_{\mathrm{upper}}}{\cos\theta_0}\sqrt{\frac{\varepsilon_{\mathrm{upper}}}{\varepsilon_{\mathrm{low}}}}\,\mathrm{mod}^2(f(l,\omega)),$$

$\mathcal{T}^*_{\omega^*}(\omega^*)$ is the maximum energy transmission factor for frequency $\omega = \omega^*$, and $\tau(\omega)$ is the weight function $[-1 \leqslant \tau(\omega) \leqslant 1]$. The set of globally optimal solutions for the optimal synthesis problem (1)–(4) is among the globally optimal solutions of the optimal synthesis problem for monochromatic action with frequency $\omega = \omega^*$. The Hamiltonian function for monochromatic action with frequency $\omega = \omega^*$ is of the form

$$H(\ ;\varepsilon)\Big|_z = \varepsilon\alpha_s(z,\omega^*) + \mu(\varepsilon)\beta_s(z,\omega^*), \quad b_{s-1} \leqslant z \leqslant b_s \quad (s = \overline{1,N}), \tag{5}$$

where b_s are the coordinates of the boundary between the layers $(s = \overline{1, N-1})$, N is the number of layers, and

$$\begin{gathered} \alpha_s(z,\omega) = -\frac{1}{\varepsilon_s}\,\mathrm{Re}\,\frac{\partial\psi_s(z,\omega)}{\partial z}\,f_s(z,\omega), \\ \beta_s(z,\omega) = \frac{1}{\mu_s}\,\mathrm{Re}\,\frac{\partial f_s(z,\omega)}{\partial z}\,\psi_s(z,\omega), \quad b_{s-1} \leqslant z \leqslant b_s \quad (s = \overline{1,N}); \end{gathered} \tag{6}$$

$\psi_s(z,\omega)$ $(b_{s-1} \leqslant z \leqslant b_s,\ s = \overline{1,N})$ is the solution of the conjugate system

$$\begin{gathered} \frac{\partial^2\psi_s(z,\omega)}{\partial z^2} + k_s^2(\omega)\psi_s(z,\omega) = 0, \quad b_{s-1} \leqslant z \leqslant b_s \quad (s = \overline{0,N+1}), \\ \psi_s(b_{s-1},\omega) = \psi_{s-1}(b_{s-1},\omega), \quad \frac{\partial\psi_s(b_{s-1},\omega)}{\partial z} = \frac{\varepsilon_s}{\varepsilon_{s-1}}\frac{\partial\psi_{s-1}(b_{s-1},\omega)}{\partial z} \quad (s = \overline{1,N+1}), \\ \frac{\partial\psi_0(0,\omega)}{\partial z} + ik_0(\omega)\psi_0(0,\omega) = 0, \quad \psi_{N+1}(l,\omega) + \frac{i}{k_{N+1}(\omega)}\frac{\partial\psi_{N+1}(l,\omega)}{\partial z} = -2\tau(\omega)\overline{f_{N+1}(l,\omega)}, \\ k_s(\omega) = \frac{\omega}{c}\sqrt{\varepsilon_s\mu_s - \varepsilon_{\mathrm{low}}\sin^2\theta_0} \quad (s = \overline{1,N}). \end{gathered} \tag{7}$$

Let N^* be the optimal number of layers, ε_s^* $(s = \overline{1,N^*})$ the optimal physical parameters of the materials of the layers, b_s^* $(s = \overline{1, N^* - 1})$ the optimal coordinates of the boundary between the layers, l^* $(l_{\min} \leqslant l^* \leqslant l_{\max})$ the optimal thickness of the system of layers ($l_{\min}$ and $l_{\max}$ the lower and upper boundaries per thickness of the system of layers).

Then, the condition

$$H(*;\varepsilon_s^*)\Big|_z = \max_{\varepsilon \in \Lambda} H(*;\varepsilon)\Big|_z, \quad b_{s-1}^* \leqslant z \leqslant b_s^* \quad (s = \overline{1, N^*}) \tag{8}$$

is satisfied.

(The omitted arguments of the Hamiltonian function are calculated for the optimal solution.)

If the optimal thickness $l^* \in (l_{\min}, l_{\max})$, then, according to [4], we obtain

$$H(*;\varepsilon_s^*)\Big|_z \equiv 0, \quad b_{s-1}^* \leqslant z \leqslant b_s^* \quad (s = \overline{1, N^*}). \tag{9}$$

One can verify that, within the sth layer, the function $\alpha_s(z,\omega)$ satisfies the following third-order differential equation:

$$\frac{\partial^3 \alpha_s(z,\omega)}{\partial z^3} + 4k_s^2(\omega)\frac{\partial \alpha_s(z,\omega)}{\partial z} = 0, \quad b_{s-1} \leqslant z \leqslant b_s \quad (s = \overline{1, N}). \tag{10}$$

The function $\beta_s(z,\omega)$ (6) also satisfies a similar equation. The general solution of Eq. (10) on the segment (b_{s-1}, b_s) is of the form

$$\begin{gathered} \alpha_s(z,\omega) = C_s(\omega)\sin(2k_s(z - b_{s-1})) + D_s(\omega)\cos(2k_s(z - b_{s-1})) + E_s(\omega), \\ b_{s-1} \leqslant z \leqslant b_s \qquad (s = \overline{1, N}), \end{gathered} \tag{11}$$

where $C_s(\omega)$, $D_s(\omega)$, and $E_s(\omega)$ are indefinite functional dependences of the frequency ω.

Using the necessary optimality conditions (8) and the form of the functions $\alpha_s(z,\omega)$ (11), one can show that, for the optimal solution, the following system holds, which describes the thicknesses and physical parameters of the layers of the optimal structure on both sides of the point of discontinuity:

$$\cot(k_s^*\Delta_s^*) = -\sigma_s^*\cot(k_{s-1}^*\Delta_{s-1}^*) + \tau_s^*\xi_{s-1}^*, \quad \xi_s^* = \tau_s^*\cot(k_{s-1}^*\Delta_{s-1}^*) - \sigma_s^*\xi_{s-1}^*. \tag{12}$$

Here

$$\sigma_s = \frac{1}{2}\left(\delta_s + \frac{1}{\delta_s}\right), \quad \tau_s = \frac{1}{2}\left(\delta_s - \frac{1}{\delta_s}\right), \quad \delta_s = \frac{Z_{s-1}}{Z_s}, \quad Z_s = \sqrt{\frac{\varepsilon_s}{\mu_s}}, \quad \xi_s = \frac{E_s}{C_s},$$

and Δ_s is the thickness of the sth layer; the subscripts s and $s-1$ refer to inner layers. Analysis of relations (12) leads to the condition $\Delta_s^* = \Delta_{s-2}^*$.

Thus, the interrelationship of parameters in the optimal structure exhibits internal symmetry. This internal symmetry property describes features of the internal relationship among groups of parameters of various types in the optimal structure: if the physical properties of the inner layers of the optimal structure are identical, their thicknesses are also identical.

We introduce a set of permittivity distributions $Q(\lambda^*)$. The elements of this set satisfy the conditions

$$\begin{gathered} 0 \leqslant \Delta_s \leqslant \frac{\lambda^*}{2\sqrt{\varepsilon_s\mu_s - \varepsilon_{\text{low}}\sin^2\theta_0}} \quad (s = \overline{1, N}), \\ \Delta_s = \Delta_{s-2} \quad (s = \overline{4, N-1}), \quad N_{\min} \leqslant N \leqslant N_{\max}, \end{gathered} \tag{13}$$

where λ^* is the wavelength ($\lambda^* = 2\pi c/\omega^*$); $[N_{\min}, N_{\max}]$ is the interval which includes the number of layers of the optimal structure. The quantities $N_{\min}$ and $N_{\max}$ can be evaluated analytically.

In addition to the set $Q(\lambda^*)$, we introduce a set of permeability distributions across the thickness of the structure $Q_0(\lambda^*)$. The elements of the set $Q_0(\lambda^*)$ satisfy the following system of relations:

$$0 \leqslant \Delta_1 \leqslant \frac{\lambda^*}{2\sqrt{\varepsilon_1\mu(\varepsilon_1) - \varepsilon_{\text{low}}\sin^2\theta_0}}, \quad 0 \leqslant \Delta_N \leqslant \frac{\lambda^*}{2\sqrt{\varepsilon_N\mu(\varepsilon_N) - \varepsilon_{\text{low}}\sin^2\theta_0}},$$

$$\Delta_s = \frac{\lambda^*}{4\sqrt{\varepsilon_s\mu_s - \varepsilon_{\text{low}}\sin^2\theta_0}} \quad (s = \overline{2, N-1}), \quad N_{\min} \leqslant N \leqslant N_{\max},$$

$$N_{\min} = \left[\frac{8\, l_{\min}\sqrt{(\bar{\varepsilon}\mu(\bar{\varepsilon}) - \varepsilon_{\text{low}}\sin^2\theta_0)(\bar{\bar{\varepsilon}}\mu(\bar{\bar{\varepsilon}}) - \varepsilon_{\text{low}}\sin^2\theta_0)}}{\lambda^*\left(\sqrt{\bar{\varepsilon}\mu(\bar{\varepsilon}) - \varepsilon_{\text{low}}\sin^2\theta_0} + \sqrt{\bar{\bar{\varepsilon}}\mu(\bar{\bar{\varepsilon}}) - \varepsilon_{\text{low}}\sin^2\theta_0}\right)}\right],$$

$$N_{\max} = \left[\frac{8\, l_{\max}\sqrt{(\bar{\varepsilon}\mu(\bar{\varepsilon}) - \varepsilon_{\text{low}}\sin^2\theta_0)(\bar{\varepsilon}\mu(\bar{\bar{\varepsilon}}) - \varepsilon_{\text{low}}\sin^2\theta_0)}}{\lambda^*\left(\sqrt{\bar{\varepsilon}\mu(\bar{\varepsilon}) - \varepsilon_{\text{low}}\sin^2\theta_0} + \sqrt{\bar{\varepsilon}\mu(\bar{\bar{\varepsilon}}) - \varepsilon_{\text{low}}\sin^2\theta_0}\right)}\right] + 4. \tag{14}$$

Here $\bar{\varepsilon}$ and $\bar{\bar{\varepsilon}}$ are the permittivities of the materials of the admissible set.

Joint analysis of the necessary optimality conditions and the properties of the boundary-value problem (1) leads to the following conclusions.

When the optimal thickness of the structure l^* is limiting, the set $Q(\lambda^*)$ contains the set of all multilayer structures that provide for a global minimum of the functional (3) in the synthesis problem (1)–(4). The optimization problem on the set $Q(\lambda^*)$ is three-parameter. Therefore, the set of all multilayer structures $E^*_{\omega^*}$ that provide for a global minimum of the quality functional in the synthesis problem (1)–(4) can be constructed fairly effectively and with high accuracy.

When the optimal thickness $l^* \in [l_{\min}, l_{\max}]$, the set $Q_0(\lambda^*)$ contains the set of all variants of multilayer structures that provide for a global minimum of the quality functional in the synthesis problem (1)–(4). Obviously, $Q_0(\lambda^*) \subset Q(\lambda^*)$, the optimization problem on the set $Q_0(\lambda^*)$ is one-parametric, and the independent control parameter will be the thickness of one of the boundary layers.

Hence, if the frequency $\omega = \omega^*$ is known, then, when the optimal thickness l^* is limiting, the initial multiparameter synthesis problem reduces to the three-parameter problem of minimization of criterion (3) on the set $Q(\lambda^*)$ (13). When the optimal thickness l^* is within the admissible interval of thicknesses, the initial multiparameter synthesis problem reduces to the one-parameter problem of minimization of criterion (3) on the set $Q_0(\lambda^*)$ (14). Thus, for this case, the results obtained allows one to completely solve the synthesis problem, because the set of all solutions that minimize the quality criterion (3) can be found fairly effectively.

Similar results are also valid for oblique incidence of an electromagnetic wave with vertical polarization on a system of magnetodielectric layers. In this case, the compact sets $Q(\lambda^*)$ and $Q_0(\lambda^*)$ completely coincide in their structure with the corresponding sets $Q(\lambda^*)$ and $Q_0(\lambda^*)$ (14) for an electromagnetic wave with horizontal polarization.

2. Normal Incidence of an Electromagnetic Wave on a System of Dielectric Layers. In the case of normal incidence of an electromagnetic wave ($\theta_0 = 0$), the set $Q(\lambda^*)$ is of the form

$$0 \leqslant \Delta_s \leqslant \frac{\lambda^*}{2\sqrt{n_s^2 - \varepsilon_{\text{low}}\sin^2\theta_0}} \qquad (s = \overline{1, N}),$$

$$\cot(k_{s+1}\Delta_{s+1}) = -\frac{n_s}{n_{s+1}}\cot(k_s\Delta_s) \qquad (s = \overline{2, N-2}), \tag{15}$$

$$\left[\frac{8\, l_{\min} n_{\min}}{\lambda^*}\right] \leqslant N \leqslant \left[\frac{8\, l_{\max} n_{\max}}{\lambda^*}\right] + 4,$$

where n_s is the refraction index of the sth layer, and $n_{\min}$ and $n_{\max}$ are the minimum and maximum refraction indices among the materials of the admissible set.

In this case, the multiparameter synthesis problem reduces to the two-parameter optimization problem. The independent variables are the thicknesses of one boundary layer and the thickness of one inner layer.

3. Oblique Incidence of an Acoustic Wave on a Multilayer Structure. We consider oblique incidence of a plane acoustic wave on a multilayer system consisting of plane layers. The layers are assumed to consist of materials in which shear waves do not propagate. Then, the propagation of acoustic waves in the layered medium can be described by the following boundary-value problem:

$$\dot{f}(z) = \rho(z)g(z), \qquad \dot{g}(z) = -\omega^2\mu[\rho(z)]f(z), \qquad 0 \leqslant z \leqslant l, \tag{16}$$

$$g(0) = \frac{ik_{\rm low}(\omega)\cos\theta_0}{\rho_{\rm low}}(2 - f(0)), \qquad g(l) = \frac{ik_{\rm upper}(\omega)\cos\theta_{\rm upper}}{\rho_{\rm upper}} f(l).$$

Here $f(z)$ $(0 \leqslant z \leqslant l)$ is the complex amplitude of the acoustic wave, $\rho(z)$ is the density distribution across the thickness of the structure $(0 \leqslant z \leqslant l)$, $k_{\rm low}(\omega) = \omega/c_{\rm low}$, $k_{\rm upper}(\omega) = \omega/c_{\rm upper}$, $\rho_{\rm low}$ and $c_{\rm low}$ are the density and the wave velocity in the half-space from which the wave comes, $\rho_{\rm upper}$ and $c_{\rm upper}$ are the density and the wave velocity in the half-space to which the wave enters after leaving the structure,

$$\cos\theta_{\rm upper} = \left(1 - \frac{c^2_{\rm upper}}{c^2_{\rm low}}\sin^2\theta_0\right)^{1/2}, \quad \mu(z) = \frac{c^{-2}(z) - c^{-2}_{\rm low}\sin^2\theta_0}{\rho(z)},$$

and $c(z)$ $(0 \leqslant z \leqslant l)$ is the velocity distribution of the acoustic wave across the thickness of the structure. The physical parameters of the layered structure are assumed to be related by the functional dependence $c = c(\rho)$, which allows one to determine unambiguously the sound velocity in the material from its density. In this case, the only independent physical parameter is the density ρ. Let the admissible set of materials Λ consist of only two materials with densities $\bar{\rho}$ and $\bar{\bar{\rho}}$. For each $z \in [0, l]$, the inclusion

$$\rho(z) \in \Lambda \tag{17}$$

holds.

The energy transmission factor for the acoustics wave is determined by solving the boundary-value problem (16) for $z = l$:

$$\mathfrak{T}(\omega) = \frac{c_{\rm low}\rho_{\rm low}\cos\theta_{\rm upper}}{c_{\rm upper}\rho_{\rm upper}\cos\theta_0}\,{\rm mod}^2 f(l, \omega).$$

It is required to design a layered structure with high reflection of acoustic waves in some spectral regions and with low reflection in other regions. In a variational formulation, this problem involves minimization of the criterion

$$J = \int_{\omega_{\min}}^{\omega_{\max}} \tau(\omega)\mathfrak{T}(\omega)\,d\omega \to \min \tag{18}$$

under the extra condition that, at frequency $\omega = \omega^*$, the functional characteristics of the structure must be maximal:

$$\mathfrak{T}(\omega^*) = \mathfrak{T}^*_{\omega^*}(\omega^*). \tag{19}$$

Here $\tau(\omega)$ is weight function $(-1 \leqslant \tau(\omega) \leqslant 1)$ and $\mathfrak{T}^*_{\omega^*}(\omega^*)$ is the maximum energy transmission factor for the frequency $\omega = \omega^*$.

We introduce a set of density distributions $Q(f^*) f^* = \omega^*/2\pi$. The elements of the set $Q(f^*)$ satisfy the relations

$$\begin{gathered} 0 \leqslant \Delta_s \leqslant \frac{1}{2f^*\sqrt{c_s^{-2} - c_{\rm low}^{-2}\sin^2\theta_0}} \qquad (s = \overline{1, N}), \\ \Delta_s = \Delta_{s-2} \quad (s = \overline{4, N-1}), \quad N_{\min} \leqslant N \leqslant N_{\max}, \end{gathered} \tag{20}$$

where $[N_{\min}, N_{\max}]$ is an interval that contains the number of layers of the optimal structure. For the quantities $N_{\min}$ and $N_{\max}$, analytical estimates can be obtained.

In addition to the set $Q(f^*)$, we introduce a set of density distributions $Q_0(f^*)$. The elements of the set $Q_0(f^*)$ satisfy the relations

$$\begin{gathered} 0 \leqslant \Delta_1 \leqslant \frac{1}{2f^*\sqrt{c_1^{-2} - c_{\rm low}^{-2}\sin^2\theta_0}}, \quad 0 \leqslant \Delta_N \leqslant \frac{1}{2f^*\sqrt{c_N^{-2} - c_{\rm low}^{-2}\sin^2\theta_0}}, \\ \Delta_s = \frac{1}{4f^*\sqrt{c_s^{-2} - c_{\rm low}^{-2}\sin^2\theta_0}} \quad (s = \overline{2, N-1}), \quad N_{\min} \leqslant N \leqslant N_{\max}, \end{gathered} \tag{21}$$

$$N_{\min} = \left[\frac{8f^* l_{\min} \sqrt{(\bar{c}^{-2} - c_{\text{low}}^{-2} \sin^2\theta_0)(\bar{\bar{c}}^{-2} - c_{\text{low}}^{-2} \sin^2\theta_0)}}{\sqrt{\bar{c}^{-2} - c_{\text{low}}^{-2} \sin^2\theta_0} + \sqrt{\bar{\bar{c}}^{-2} - c_{\text{low}}^{-2} \sin^2\theta_0}} \right],$$

$$N_{\max} = \left[\frac{8f^* l_{\max} \sqrt{(\bar{c}^{-2} - c_{\text{low}}^{-2} \sin^2\theta_0)(\bar{\bar{c}}^{-2} - c_{\text{low}}^{-2} \sin^2\theta_0)}}{\sqrt{\bar{c}^{-2} - c_{\text{low}}^{-2} \sin^2\theta_0} + \sqrt{\bar{\bar{c}}^{-2} - c_{\text{low}}^{-2} \sin^2\theta_0}} \right] + 4.$$

Here $\bar{c} = c(\bar{\rho})$ and $\bar{\bar{c}} = c(\bar{\bar{\rho}})$.

Joint analysis of the necessary optimality conditions and the properties of the boundary-value problem (16) leads to the following conclusions.

When the optimal thickness of the structure l^* is limiting, the set $Q(f^*)$ contains the set of all multilayer structures that provide for a global minimum of the quality functional (18) in the synthesis problem (16)–(19). The optimization problem on the set $Q(f^*)$ is three-parametric. Therefore, one can construct fairly effectively and with high accuracy the set of all multilayer structures $E^*_{\omega^*}$ that leads to a global minimum of the quality functional in the synthesis problem (16)–(19).

When the optimal thickness $l^* \in (l_{\min}, l_{\max})$ is limiting, the set $Q_0(f^*)$ contains the set of all multilayer structures that provide for a global minimum of the quality functional (18) in the synthesis problem (16)–(19). In this case, $Q_0(f^*) \subset Q(f^*)$. The optimization problem on the set $Q_0(f^*)$ is one-parametric, and the independent control parameter is the thickness of one of the boundary layers.

Thus, if the frequency $\omega = \omega^*$ is known, then when the optimal thickness l^* is limiting, the initial multiparameter synthesis problem reduces to the three-parameter problem of minimization of criterion (18) on the set $Q(f^*)$ (20). If $l^* \in (l_{\min}, l_{\max})$, the initial multiparameter synthesis problem reduces to the one-parameter problem of minimization of criterion (18) on the set $Q_0(f^*)$ (21). For this case, the result obtained allows one to completely solve the synthesis problem, because the set of all solutions that provide for a global minimum of the quality criterion (10) can be found effectively.

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